Homogeneous coordinate rings

Let $\emptyset \neq V \subseteq \mathbb{P}^n$ be a variety

Def: $\Gamma_h(V) = k[x_1, \dots, x_{n+1}] / I_p(V)$ is the homogeneous coordinate ring of V.

Caution: The elements of $\Gamma_h(V)$ are not functions on V. In fact, homogeneous polynomials aren't even functions on $\mathbb{P}^n$!

e.g. $f = x+y$, $f([0:1]) \neq f([0:2])$.

Def: Let $I \subseteq k[x_1, \dots x_{n+1}]$ be a homogeneous ideal (not necessarily prime) and let $\Gamma = k[x_1, \dots, x_{n+1}] / I$.

$f \in \Gamma$ is a form of degree d if there is a form $F \in k[x_1, \dots, x_{n+1}]$ s.t. $\overline{F} = f$ in Γ.

Remark: This degree is well-defined: Suppose F and G are forms and $\overline{F} = \overline{G}$ in Γ. Then $F - G \in I$.

If $\deg F \neq \deg G$, then $F, G \in I$ since I is homogeneous, so $\overline{F} = \overline{G} = \overline{0}$.

Prop: Every $f \in \Gamma$ may be written uniquely as $f = f_0 + \dots + f_d$ w/ f_i a form of degree i.

Pf: If $g \in k[x_1, \dots, x_{n+1}]$ s.t. $\overline{g} = f$, we can write $g = \sum g_i$, so $f = \sum \overline{g_i}$.

For uniqueness, assume $f = \sum \overline{h_i}$, where $h_i \in k[x_1, \ldots, x_{n+1}]$ is a form of degree i. Then $\sum (g_i - h_i) \in I \Rightarrow g_i - h_i \in I \Rightarrow \overline{g_i} = \overline{h_i}$. $\square$

Rational functions

Let $k_h(V)$ be the field of fractions of $\Gamma_h(V)$, called the homogeneous function field of V.

Note: Unlike in the affine case, most elements of $k_h(V)$ are not functions on any subset of V.

However, if $P = [a_1 : \ldots : a_{n+1}] \in \mathbb{P}^n$ and F and G are forms of degree d s.t. $G(P) \neq 0$, then

$$\frac{F(\lambda a_1, \ldots, \lambda a_{n+1})}{G(\lambda a_1, \ldots, \lambda a_{n+1})} = \frac{\lambda^d F(a_1, \ldots, a_{n+1})}{\lambda^d G(a_1, \ldots, a_{n+1})},$$ so F/G is well-defined

at P in this case!

Def: The field of rational functions on V is

$$k(V) = \left\{ z \in k_h(V) \;\middle|\; z = F/G,\ F, G \in \Gamma_h(V) \text{ forms of the same degree} \right\}$$

Elements of $k(V)$ are called rational functions on V.

Check: $k(V)$ is a subfield of $k_h(V)$.

Note: $k \subseteq k(V) \subseteq k_h(V)$ but $\Gamma_h(V) \nsubseteq k(V)$

Ex: Consider $\underset{\{[1:y]\}}{\underset{\shortparallel}{U_1}} \cong \mathbb{A}^1 \subseteq \underset{\{[x:y]\}}{\underset{\shortparallel}{\mathbb{P}^1}}$

If $z = \frac{F}{G} \in k(\mathbb{P}^1)$, then $\frac{F(1,y)}{G(1,y)} \in k(U_1)$.

In fact, every function in $k(U_1)$ can be written in this way (see HW), so $k(\mathbb{P}^1) \cong k(\mathbb{A}^1)$.

Def: Let $P \in V$, $\alpha \in k(V)$. α is <u>defined</u> at P if $\alpha = \frac{F}{G}$ forms of same deg s.t. $G(P) \neq 0$.

The <u>local ring</u> of V at P is $\mathcal{O}_P(V) = \{\alpha \in k(V) \mid \alpha \text{ is defined at } P\}$

Note: $\mathcal{O}_P(V)$ is a subring of $k(V)$, and it is in fact local with maximal ideal

$$m_P(V) = \left\{\frac{F}{G} \,\middle|\, G(P) \neq 0,\ F(P) = 0\right\} \quad \text{(exer)}$$

Ex: Let $P = [0:0:1] \in \mathbb{P}^2$. Then

$$\mathcal{O}_P(\mathbb{P}^2) = \left\{\frac{F}{G} \,\middle|\, G(P) \neq 0\right\} = \left\{\frac{F}{H + z^d} \,\middle|\, \begin{array}{l} F, H \text{ forms} \\ \text{of deg } d,\ H \in (x,y) \end{array}\right\}$$

$$m_P(\mathbb{P}^2) = \left\{\frac{F}{H + z^d} \,\middle|\, F, H \in (x,y)\right\}$$

Consider $U_3 \subseteq \mathbb{P}^3$. Then

$$U_z \longleftrightarrow \mathbb{A}^2$$
$$[a:b:1] \leftrightarrow (a,b)$$
$$P \leftrightarrow (0,0)$$

Let $\varphi : \mathcal{O}_P(\mathbb{P}^2) \longrightarrow \mathcal{O}_0(\mathbb{A}^2)$ be defined

$$\frac{F}{G+z^d} \longmapsto \frac{F(x,y,1)}{G(x,y,1)+1} \qquad \text{(quotient by } (z-1))$$

$\frac{F}{G+z^d}$ is in the kernel $\Longleftrightarrow F(x,y,1)=0 \Longleftrightarrow F \in (z-1)$

But F is homogeneous, so this can only happen if $F=0$.

Similarly, any function in $\mathcal{O}_0(\mathbb{A}^2)$ arises this way (see hw)

so φ is an isomorphism.